Page From a Physicist's Notebook

Momentum 103

The algebraic solution for conservation of momentum

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This *Page from a Physicist's Notebook* is the third in a series of related articles on the topic of momentum.

In the first article in the series (*Momentum 101 – The principle of conservation of linear momentum*) we used Newton's laws of motion, and one of the equations for uniform motion, to develop the equation for the conservation of linear momentum. [1]

The second article (*Momentum 102 – Vector Analysis and Momentum*) implemented a graphical solution to the equation for the conservation of linear momentum. [2] In particular, we used a vector diagram, drawn to scale, to obtain the solution for the initial speeds of two-vehicles involved in an angled collision.

In the current article, we will consider a general form of the vector diagram for a two-vehicle collision, and develop a purely algebraic solution.

From basic principles…

Conservation of linear momentum

$$\overrightarrow{m_1V_1} + \overrightarrow{m_2V_2} = \overrightarrow{m_1V_1'} + \overrightarrow{m_2V_2'}$$

Trigonometrical functions:

$$\sin\theta = \frac{\text{Opposite}}{\text{Hypotenuse}}$$

$$\cos\theta = \frac{\text{Adjacent}}{\text{Hypotenuse}}$$

Vector Diagram

While the vector diagram will be general in nature, note that it is always necessary to select a datum line (a zero-degree reference direction) for the measurement of the vehicle approach and departure angles.

Since this reference direction is arbitrary, we can make our lives a little simpler by opting to set the datum line along the initial direction of travel of Vehicle 1, i.e. the approach angle for Vehicle 1 will be 0º.

Adopting this convention, the vector parallelogram for a two-vehicle collision is as shown in Figure 1. Note that the two final momentum vectors, AB and AD (m_1V_1' and m_2V_2') add together to give the resultant vector AC (R). Similarly, and in order to satisfy the principle of conservation of momentum, the two initial momentum

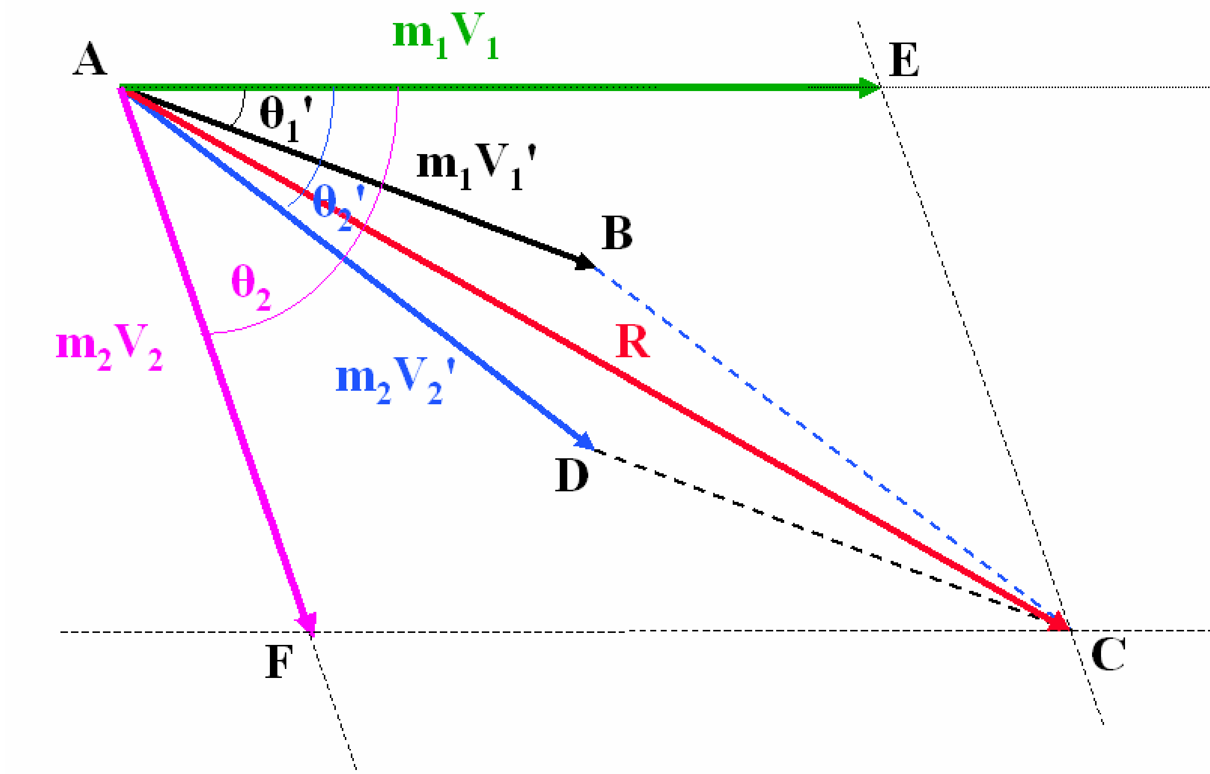

Figure 1 Vector parallelogram for a two-vehicle collision

vectors, AE and AF (m_1V_1 and m_2V_2) also add together to give the resultant vector AC.

This is essentially the vector diagram that we used previously (*Momentum 102 – Vector Analysis and Momentum*) to develop a graphical solution of the momentum equation and to derive the two unknown initial vehicle speeds (V_1 and V_2).

In Figure 1, the parallelogram ABCD is used to add the final momentum vectors together to give the resultant, while the parallelogram AECF adds the initial momentum vectors to give the same resultant vector.

We can use the property of vector equality to identify that a vector drawn as the line BC is the same as the vector AD (i.e. the final momentum of Vehicle 2, m_2V_2'). The line BC has the same length as AD, and points in the same direction (i.e. at the departure angle for Vehicle 2). Thus, we can also represent the final momentum of Vehicle 2 as the line BC. We would then have the triangle ABC being used to add the two final momentum vectors to give the resultant vector AC.

Similarly, the vector EC is the same as the vector AF, the initial momentum of Vehicle 2 (m_2V_2). Thus, vector triangle AEC could be used to add the two initial momentum vectors to give the resultant AC.

The vector diagram, revised to use the two vector triangles, ABC and AEC, is shown in Figure 2.

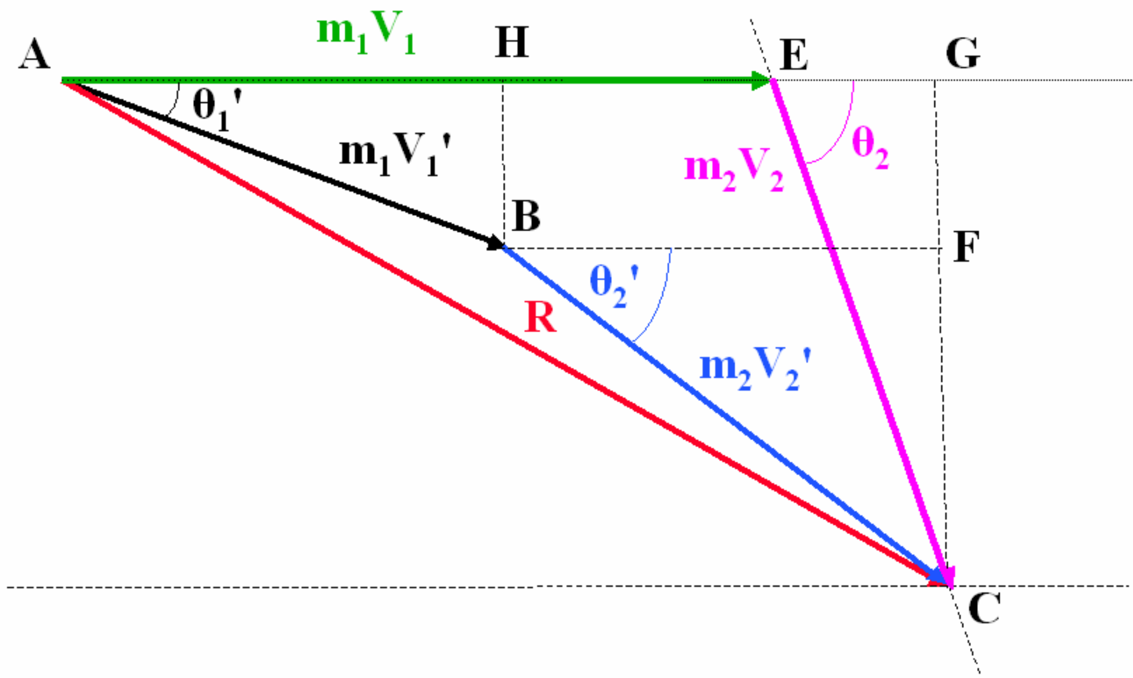

Figure 2 Vector triangles for a two-vehicle collision

Note that, in Figure 2, we have the two final momentum vectors AB and BC (m_1V_1' and m_2V_2') adding together to give the resultant vector AC, and the two initial momentum vectors AE and EC (m_1V_1 and m_2V_2) adding to give the same resultant AC.

By adding a few construction lines we can identify some right-angled triangles that are associated with the vector diagram, notably AGC, AHB, EGC and BFC.

But, why would we do that?

The rationale for doing this may not be immediately apparent. But, recall that our present aim is to develop an algebraic solution to the conservation of momentum equation. You may also remember that in the second article of this series, we noted that the solution to the single equation in which there were two unknowns was only possible because we were able to work in two dimensions (x and y), thus taking into account both the magnitudes and directions of the momentum vectors.

The algebraic solution requires the same process. We need to develop two equations, one along the zero-degree reference line (effectively our x-axis), and one in a perpendicular direction (our y-axis).

These will provide two simultaneous equations for the two unknown vehicle speeds (V_1 and V_2) from which we will be able to derive the required solutions.

What we are going to do is look at the lines AG and GC, and see how the lengths of these lines are made up from components of various momentum vectors. To do this we will need to apply our knowledge of the sines and cosines of angles in right-angled triangles.

Initially, we are going to consider the length of the line GC. Looking at triangle EGC, the sine of angle θ_2 is defined as:

$$\sin\theta_2 = \frac{GC}{EC} \quad \text{(opposite/hypotenuse)}$$

so that:

$$GC = EC \sin\theta_2$$

But, the length of EC is just the magnitude of the initial momentum of Vehicle 2 (m_2V_2), so that:

$$GC = m_2V_2 \sin\theta_2 \qquad (1)$$

We can also consider the length of this line to be made up in a different way. In particular, it is the sum of the lengths of the lines HB (the same as GF) and FC. Thus, in equation 1:

$$GC = m_2V_2 \sin\theta_2 = HB + FC$$

$$m_2V_2 \sin\theta_2 = HB + FC \qquad (2)$$

Now, in triangle AHB:

$$\sin\theta_1' = \frac{HB}{AB} \quad \text{(opposite/hypotenuse)}$$

$$HB = AB \sin\theta_1'$$

where AB is the final momentum of Vehicle 1 (m_1V_1') so that:

$$HB = m_1V_1' \sin\theta_1' \qquad (3)$$

Similarly, in triangle BFC:

$$\sin\theta_2' = \frac{FC}{BC} \quad \text{(opposite/hypotenuse)}$$

$$FC = BC \sin\theta_2'$$

where BC is the final momentum of Vehicle 2 (m_2V_2') so that:

$$FC = m_2V_2' \sin\theta_2' \qquad (4)$$

Substituting equations 3 and 4 into equation 2 gives:

$$m_2V_2 \sin\theta_2 = m_1V_1' \sin\theta_1' + m_2V_2' \sin\theta_2'$$

Hence:

$$V_2 = \frac{m_1V_1' \sin\theta_1' + m_2V_2' \sin\theta_2'}{m_2 \sin\theta_2} \qquad (5)$$

Equation 5 provides a method of calculating V_2, the initial speed of Vehicle 2, from the masses, run-out speeds, and departure angles of the two vehicles, together with the approach angle of Vehicle 2.

Half-way there !

We can calculate V_2 using Equation 5

Now, let's take a look at the length of the line AG in Figure 2 and adopt similar techniques to those used above. Note that we can consider the length of this line to be made up in two ways. It is the sum of the lengths of lines AE and EG. It is also the sum of the lengths of lines AH and BF (the same as HG).

Thus:

$$AE + EG = AH + BF \qquad (6)$$

Now, the length of AE is just the magnitude of the initial momentum of Vehicle 1 (m_1V_1) so that:

$$AE = m_1V_1 \qquad (7)$$

In addition, EGC is a right-angled triangle where the cosine of θ_2 is:

$$\cos\theta_2 = \frac{EG}{EC} \quad \text{(adjacent/hypotenuse)}$$

and so:

$$EG = EC\cos\theta_2$$

But, EC is the magnitude of the initial momentum of Vehicle 2 (m_2V_2), so that:

$$EG = m_2V_2\cos\theta_2 \qquad (8)$$

Substituting equations 7 and 8 into equation 6 gives:

$$m_1V_1 + m_2V_2\cos\theta_2 = AH + BF \qquad (9)$$

Looking at triangle AHB:

$$\cos\theta_1' = \frac{AH}{AB}$$

$$AH = AB\cos\theta_1'$$

where $AB = m_1V_1'$ so that:

$$AH = m_1V_1'\cos\theta_1' \qquad (10)$$

Similarly, in triangle BFC:

$$\cos\theta_2' = \frac{BF}{BC}$$

$$BF = BC\cos\theta_2'$$

where $BC = m_2V_2'$ so that:

$$BF = m_2V_2'\cos\theta_2' \qquad (11)$$

Substituting equations 10 and 11 into equation 9 gives:

$$m_1V_1 + m_2V_2\cos\theta_2 = m_1V_1'\cos\theta_1' + m_2V_2'\cos\theta_2'$$

It follows that:

$$m_1V_1 = m_1V_1'\cos\theta_1' + m_2V_2'\cos\theta_2' - m_2V_2\cos\theta_2$$

$$V_1 = \frac{m_1V_1'\cos\theta_1' + m_2V_2'\cos\theta_2' - m_2V_2\cos\theta_2}{m_1} \qquad (12)$$

Wow! That's a lot of mathematics. But, note that we really only have to understand how to use the sine and cosine of an angle. Everything else is simple addition, subtraction, multiplication and division.

So, once we have obtained the initial speed of Vehicle 2 (V_2) from equation 5, we can then use equation 12 to calculate the initial speed of Vehicle 1 (V_1) since we know the masses, run-out speeds, and departure angles for both vehicles, and we also know the approach angle for Vehicle 2.

Conservation of linear momentum

$$V_2 = \frac{m_1 V_1' \sin \theta_1' + m_2 V_2' \sin \theta_2'}{m_2 \sin \theta_2}$$

$$V_1 = \frac{m_1 V_1' \cos \theta_1' + m_2 V_2' \cos \theta_2' - m_2 V_2 \cos \theta_2}{m_1}$$

Case Study

In the second article in this series we analyzed a real-world collision using the graphical solution for the conservation of linear momentum. The crash involved the front of a 2000 Chevrolet Impala four-door sedan (Vehicle 1) striking the left side of a 1998 Saturn SL2 (Vehicle 2) as the two vehicles travelled into a four-leg intersection (Figure 3).

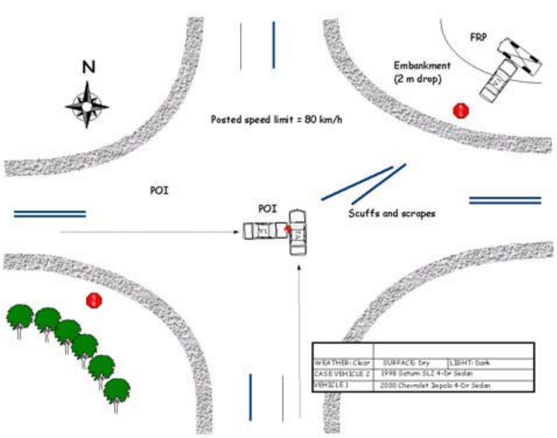

Figure 3 Collision Schematic

Recall that, for this case study, we selected due east as the zero-degree reference line for the measurement of the approach and departure angles of the vehicles, i.e. our datum is along the initial travel direction for Vehicle 1 ($\theta_1 = 0°$).

This measurement convention is the same as that used to derive the above-equations. We can, therefore, apply these equations to our real-world crash in order to determine the two initial vehicle speeds, V_1 and V_2.

The data obtained from the subject collision investigation were as follows:

Vehicle 1 (Chevrolet Impala)

- $m_1 = 1710$ kg
- $V_1' = 51$ km/h
- $\theta_1' = 329°$
- $\theta_1 = 0°$

Vehicle 2 (Saturn SL2)

- $m_2 = 1160$ kg
- $V_2' = 51$ km/h
- $\theta_2' = 321°$
- $\theta_2 = 270°$

Substituting these values into equation 5, allows us to calculate the initial speed of Vehicle 2:

$$V_2 = \frac{m_1 V_1' \sin \theta_1' + m_2 V_2' \sin \theta_2'}{m_2 \sin \theta_2}$$

$$V_2 = \frac{1710 x 51 x \sin(329°) + 1160 x 51 x \sin(321°)}{1160 x \sin(270°)}$$

$$V_2 = \frac{87210 x (-0.5150) + 59160 x (-0.6293)}{1160 x (-1)}$$

$$V_2 = \frac{-44916.5 - 37230.6}{-1160}$$

$$V_2 = \frac{-82147.1}{-1160}$$

V_2= 71 km/h

[Note that the sine of any angle in the range 270 through 360 is negative. We end up with -82147 being divided by -1160, the result being +71 km/h since effectively we have: (-1 x 82147)/(-1 x 1160) and the two -1's cancel out.]

Now that we have calculated V_2, we can use equation 12 to determine V_1

$$V_1 = \frac{m_1 V_1' \cos \theta_1' + m_2 V_2' \cos \theta_2' - m_2 V_2 \cos \theta_2}{m_1}$$

V_1 = { 1710x51xcos(329°) +
1160x51xcos(321º) -
1160x71xcos(270º) }
/ 1710

V_1 = { 87210x0.8572 + 59160x0.7771
- 82360x0 } / 1710

V_1 = { 74753.6 + 45976.0 -0) }/1710

V_1 = {120729.6 }/1710

V_1 = 71 km/h

So, our algebraic solution gives the initial speeds of the vehicles as:

V_1 = 71 km/h

V_2= 71 km/h

As might be expected, these are precisely the same values obtained from the graphical solution.

Graphical vs. Numerical Analysis

We have now seen that the graphical and algebraic solutions to the conservation of linear momentum both provide the same answers.

We noted previously that a vector diagram, drawn to scale, provides a very direct representation of the subject collision such that it is difficult to make a mistake in the process of obtaining a solution using the graphical method.

In contrast, the mathematical equations we have developed here will give answers based on whatever data we input. It should be obvious that, given the complexity of the equations, and the strict convention used to determine the various angles involved, we need to take considerable care with the algebraic solution. In particular, we need to ensure that the approach and departure angles are identified and measured correctly. We must pay careful attention to the individual parameters in the different terms of the equations in order that the correct term is matched to the correct piece of data.

Furthermore, some care must be taken with the signs (especially the negative sign in the last term of the numerator in equation 12).

However, the good news is that, if we strictly adhere to the convention for the measurement of the approach and departure angles (0-360 degrees from the datum), the equations will automatically take care of whether the sines and cosines of the angles are positive or negative.

And, the really good news is that the algebraic method, because it is based on mathematical formulae, lends itself to a computer-based solution.

Computer Spreadsheet

Spreadsheet programs, such as Microsoft Excel [1], Lotus 1-2-3 [2], and Quattro Pro [3], provide us with the ability to enter formulae into individual cells of the spreadsheet.

It has been my experience to date that many police officers are unfamiliar with the use of spreadsheet programs. This is unfortunate, since spreadsheets provide a flexible platform for easily conducting a wide range of collision-related calculations. Furthermore, because of their ability to almost instantly recalculate values based on changed parameters, they lend themselves exceptionally well to "what-if" calculations, uncertainty estimation, and sensitivity analysis using bracketed values of measured quantities. In this section, we will explore in detail a specific spreadsheet designed to perform the calculations required for the momentum analysis of a two-vehicle crash.

[1] http://en.wikipedia.org/wiki/Microsoft_Excel
[2] http://en.wikipedia.org/wiki/Lotus_1-2-3
[3] http://en.wikipedia.org/wiki/Quattro_Pro

The commands used in spreadsheet formulae can be very simple, such as adding the contents of a number of cells to provide a total, as is frequently used in financial accounting. But, the programs are quite sophisticated so that, in addition to performing simple mathematical calculations, they have many built-in functions, such as square roots, sines and cosines. Furthermore, one can set up certain cells to accept basic data, and place formulae in other cells to compute both intermediate and final results.

As noted earlier, the really good news is that if we enter different values into the cells containing the base elements, the program rapidly updates the entire spreadsheet, performing all the necessary calculations, and determining the new values of all the cells containing formulae.

Since spreadsheets know about addition, subtraction multiplication and division, and even square roots, sines and cosines, they are great tools for calculating initial vehicle speeds using the equations for the conservation of linear momentum that we have developed above.

An example of such a spreadsheet, set up using Microsoft Excel, is shown in Figure 4. Note that certain cells are highlighted in light grey to indicate that these cells are for the user to input data. For ease of reference, the columns in spreadsheets are identified with alphabetic characters (A, B, C…) while the rows are numbered (1, 2, 3…)

The spreadsheet is designed for general use. Cells B3 through B5 accept text strings designating the name of the recon-structionist, the case number, and a date. This allows us to customize each version of the spreadsheet with specific information

Microsoft Excel - Momentum.xls

	A	B	C	D	E	F	G	H
1	**Conservation of Linear Momentum**							
2								
3	**Analyst**	**Alan German**						
4	**Case No.**	**Momentum 103**						
5	**Date**	**19-Dec-06**						
6								
7			**Vehicle 1**			**Vehicle 2**		
8								
9		Type	2000 Chevrolet Impala			1998 Saturn SL2		
10								
11		Mass (kg)	1710			1160		
12								
13		Coefficient of friction	0.4			0.4		
14		Stopping distance (m)	26			26		
15								
16		Departure angle (degrees)	329			321		
17		Approach angle (degrees)	**0**			270		
18								
19		Final velocity	51.28	km/h		51.28	km/h	
20			@	329 degrees		@	321 degrees	
21								
22		Initial velocity	70.98	km/h		71.20	km/h	
23			@	000 degrees		@	270 degrees	
24								
25	**Note:**							
26								
27	For Vehicle 1:							
28								
29	Final velocity	=15.9*SQRT(C13*C14)						
30								
31	Initial velocity	=(C11*C19*COS(RADIANS(C16))+F11*F19*COS(RADIANS(F16))-F11*F22*COS(RADIANS(F17)))/C11						
32								
33	For Vehicle 2:							
34								
35	Final velocity	=15.9*SQRT(F13*F14)						
36								
37	Initial velocity	=(C11*C19*SIN(RADIANS(C16))+F11*F19*SIN(RADIANS(F16)))/(F11*SIN(RADIANS(F17)))						
38								
39								
40								

Momentum

Ready

Figure 4. Computer spreadsheet for conservation of momentum calculations

about the subject collision and makes the spreadsheet self-documenting.

The next set of rows in the spreadsheet (Rows 9-17) are used to accept the crash data – the vehicle identifier (year, make and model), vehicle mass, coefficient of friction, stopping distance, departure and approach angles – for each of the involved vehicles. Note that the units to be used for these parameters are specified where appropriate.

For example, the vehicle stopping distance is to be entered in metres, since the subsequent calculation of run-out speed ($V=15.9\sqrt{\mu d}$) takes the form to calculate speed in km/h if the stopping distance is measured in metres. Similarly, the approach and departure angles are to be specified in degrees. But, take particular note that the approach angle for Vehicle 1 is not to be entered. This value (in cell C17) is set to zero since this was the basic assumption used to derive our momentum equations.

Row 19 contains two cells with the first calculated results. Cells C19 and F19 compute the run-out speeds of Vehicle 1 and Vehicle 2 respectively. The note on row 29 shows the formula that the spreadsheet uses for the calculation of the magnitude of the final velocity of Vehicle 1. The spreadsheet uses the square root function in the form:

V_1 = 15.9*SQRT(C13*C14)

Compare this to our slide-to-stop equation:

$$V_1 = 15.9\sqrt{\mu d}$$

In the spreadsheet's formula, cell C13 references the value of the coefficient of friction that applies to the run-out motion of Vehicle 1, while C14 is the vehicle's stopping distance from the point of impact to its final resting position. The asterisk indicates that multiplication is to be performed, while SQRT is the square root function that applies to the terms inside the brackets.

Cell F22 contains the formula to calculate the initial speed of Vehicle 2. This is essentially our equation 5. The actual formula used in the spreadsheet is shown in the note on row 37. Note that there are multiple sets of brackets to ensure that the various parameters are processed in the correct order. Careful inspection of this spreadsheet formula will show it to be precisely the same as equation 5. For example, the first term in equation 5 is:

$$m_1 V_1' \sin \theta_1'$$

while the first term to be evaluated in cell F22 of the spreadsheet is:

C11*C19*SIN(RADIANS(C16))

By cross-referencing the terms, we can see that:

Cell C11 references m_1, the mass of Vehicle 1.

Cell C19 references V_1', the calculated run-out speed of Vehicle 1.

Cell C16 references θ_1', the departure angle for Vehicle 1.

But, note that the angle, measured in degrees, is converted to radians using the RADIANS function before the sine of the angle is computed.

$\theta_1' = 329^o$

RADIANS(329º) = 5.742 radians

This is necessary because, in Excel, the SIN function takes the angle argument in radians rather than in degrees.

SIN(5.742 radians) = -0.515

Thus, we can see that the first term of the formula used in the spreadsheet is precisely equivalent to the first term of equation 5. Similarly, we could show that all of the subsequent terms are precisely matched so that the formula used in cell F22 is a true representation of equation 5. Thus, cell F22 will calculate the initial speed of Vehicle 2.

Through a similar process to that noted above, we could show that the formula used in cell C22 is equivalent to equation 12 and hence calculates the initial speed of Vehicle 1.

Note that the cells C22 and F22 are labeled (cell B22) as "Initial velocity". Since we are using conservation of momentum, and conducting vector analysis, it is appropriate to use this terminology. However, if we state that we are providing velocity, we must give both the speeds (the magnitudes of the velocity vectors) and the associated directions.

In consequence, the spreadsheet notes both the initial speed of Vehicle 2 (cell F22) and its direction (cells F23 and G23). The direction is given as being "@ 270 degrees" (to the zero-degree reference line). This is achieved simply by using an @ symbol in F23, and by restating the vehicle's approach angle in G23 using the formula:

=TEXT(F17,"000")&" degrees"

The TEXT function is first used to convert the number of degrees for the approach angle to a three-character string ("270"). The concatenation operator (&) is then used to combine this text string with a space and the word degrees

A similar procedure is used to specify "@ 000 degrees" in cells C23 and D23 as the approach direction for Vehicle 1.

So, we now have a fully-developed spreadsheet that will calculate the run-out speeds of the vehicles based on the coefficients of friction and stopping distances provided, and then go on to compute the two initial vehicle speeds based on vector analysis of the momentum diagram. The spreadsheet is essentially self-documenting since we can include specific information to identify a particular case, and each of the input data fields is labeled, as are the cells containing the calculated values. In addition, the formulae used for the numerical calculations are included on the spreadsheet in the "notes" section.

Note that we can readily reuse the spreadsheet, either by copying it to a new file and then modifying the latter to fit a different case, or by copying it to a new worksheet within the spreadsheet, and maintaining separate worksheets for different cases.

Creating a new worksheet, based on the initial worksheet, is achieved by right-clicking the mouse on the "Momentum" tab at the bottom of the sheet, and selecting "Move or Copy". Choose "Move to end" and click on the checkbox marked "Create a copy". This generates a second worksheet, named "Momentum (2)". This sheet can be renamed by simply right-clicking on its tab and selecting the "Rename" option. The second worksheet can then be modified by changing the identifiers and input variables to suit a new collision scenario.

In-line Collisions

If we try to use the spreadsheet (or try to manually solve equations 5 and 12) when both approach angles are zero, we will encounter a #DIV/0! error. This indicates that we are trying to divide by zero which is not allowed since it would generate an infinite value. Clearly, the input parameters are in violation of the assumptions used for our momentum analysis.

The reason is clear; we have an “in-line” collision, normally either a head-on or a rear-end impact, where both vehicles are initially moving along the same straight line. Since all the initial momentum is directed along the x-axis, all the final momentum must also be directed along the x-axis. With only one direction to work with, it is not possible to solve the momentum equation for two unknown initial velocities. We need either a second equation, or we need to know one of the initial vehicle speeds.

Two equations can be obtained by considering both conservation of momentum and conservation of energy as these apply to a given collision. However, the latter requires accounting for the energy used in crushing the vehicle structures, a topic that is worthy of another page (or two) from a physicist’s notebook!

Knowledge of one of the initial vehicle speeds may come from pre-crash data captured by an on-board event data recorder (EDR), or from a reliable witness to the motion of one of the vehicles prior to the collision. If such an estimate is available, momentum can be used to determine the initial speed of the partner vehicle in the collision.

Another, not-uncommon situation, provides a special case where a simple formulation of conservation of momentum can be used. This is the case where one vehicle is stationary prior to the collision. Typically, this will occur in a rear-end crash where one vehicle is stopped at a traffic light or a stop sign. Sometimes, a good approximation of one vehicle having no initial velocity can be the case in side-impacts where one vehicle commences a left turn, or has just pulled out from an intersecting road, and enters the path of an on-coming vehicle.

The basic equation for conservation of momentum is:

$$m_1V_1 + m_2V_2 = m_1V_1' + m_2V_2' \qquad (13)$$

If we consider that V_2, the initial speed of Vehicle 2, is zero, this equation reduces to:

$$m_1V_1 = m_1V_1' + m_2V_2'$$

so that:

$$V_1 = \frac{m_1V_1' + m_2V_2'}{m_1} \qquad (14)$$

Frequently, in such cases, the two vehicles reach a common velocity immediately after impact. This is particularly obvious when the two vehicles become locked together on impact, and remain engaged as they move to the final rest position. If we denote the common, post-impact velocity as Vc, then:

$$V_1' = V_2' = Vc$$

and equation 14 becomes:

$$V_1 = \frac{(m_1 + m_2)\,Vc}{m_1} \qquad (15)$$

In this special case, where the initial speed of Vehicle 2 is zero, and the two vehicles both have the same post-impact speed of Vc, we can see that the calculation for the initial speed of Vehicle 1 is based on a simple mass ratio.

Conclusion

This article has provided a rigorous mathematical analysis of a generic momentum diagram for a two-vehicle, angled collision and has developed equations to calculate the two unknown initial vehicle speeds.

While these equations can be readily solved using a calculator, we have seen that a computer spreadsheet provides a simple method of performing the momentum calculations for any given case. The same spreadsheet can also be readily adapted for similar cases.

It should also be evident that we could use similar techniques to those described to develop other spreadsheets for more complex situations, for example, where a vehicle slides to a stop over multiple surfaces (different friction coefficients), or where braking efficiency terms must be introduced for non-locked wheel braking.

Note that the equations developed here to solve momentum problems are predicated on vehicle momenta occurring in two dimensions. In particular, our solution, based on Figure 2, effectively considered the total momentum of the two vehicles along the x-axis (vector AG), and the total momentum of the two vehicles along the y-axis (vector GC), as these combined to produce the total resultant momentum (vector AC). It was our ability to solve the vector equation in two dimensions that provided values for the two unknown, initial vehicle speeds.

Where the vehicle momenta act along the same straight line (e.g. along the x-axis) we have an in-line collision. In such a case, the equations developed for a two-dimensional problem are no longer appropriate and a different methodology must be applied.

In the special case where one vehicle is initially stationary, and both vehicles reach a common post-impact speed, we have seen that a simple mathematical formulation of the equation for conservation of momentum can be applied.

In other, more general situations, if an estimate of one vehicle's speed is available from some source, momentum alone (equation 13) may be used to obtain the speed of the second vehicle. Otherwise, the equation for the conservation of momentum must be combined with the equation for the conservation of energy in order to obtain both vehicle speeds. But, as noted earlier, the latter is another topic for a page from a physicist's notebook!

References

1. Momentum 101 – The Principle of Conservation of Linear Momentum; German A; EOTIS Newsletter; Vol. 4, No.2; pp. 3-11; December, 2006

2. Momentum 102 – Vector Analysis and Momentum; German A; EOTIS Newsletter; Vol. 4, No.2; pp. 12-24; December, 2006

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